

# A Critical Reassessment of $Q_7$ and $Q_8$ Matrix Elements

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## Abstract

We compare recent theoretical determinations of weak matrix elements of the electroweak penguin operators  $Q_7$  and  $Q_8$ . We pay special attention to the renormalization scheme dependence of these determinations as well as to the influence of higher dimension operators in the different approaches.

## 1 Introduction

There has been recent progress in understanding the bosonization of some of the four-quark operators which appear in the effective Lagrangian of the Standard Model describing  $K$  physics<sup>1</sup>. The bosonization of the electroweak penguin operators

$$Q_7 = 6(\bar{s}_L \gamma^\mu d_L) \sum_{q=u,d,s} e_q (\bar{q}_R \gamma_\mu q_R) \quad \text{and} \quad Q_8 = -12 \sum_{q=u,d,s} e_q (\bar{s}_L q_R) (\bar{q}_R d_L) \quad (1.1)$$

in particular, where  $e_q$  denote quark charges in units of the electric charge and summation over quark colour indices within brackets is understood, has been obtained by two different analytic methods. One of the methods uses the framework of the  $1/N_c$  expansion [3]; the other one combines dispersion relations with phenomenological input [4, 5]. Both methods are well rooted in the underlying QCD theory, and therefore they are competitive with more standard non-perturbative techniques based on lattice QCD simulations. In fact, the existence of lattice QCD estimates of matrix elements of the  $Q_7$  and  $Q_8$  operators [6, 7, 8, 9] allows for a detailed comparison between these different approaches. It is precisely this comparison which is the main concern of this letter. We shall also discuss, within the particular case of matrix elements of the  $Q_7$  operator, the rôle of “higher dimension operators” in weak matrix elements estimates which has been recently raised in refs. [10, 11].

The operator  $Q_7$  emerges at the  $M_W$  scale from considering the so-called electroweak penguin diagrams. In the presence of the strong interactions, the renormalization group evolution of  $Q_7$  from the scale  $M_W$  down to a scale  $\mu \lesssim m_c$  mixes this operator with the four-quark density-density operator  $Q_8$ , among others. The  $Q_8$  operator plays an important rôle in the phenomenology of CP violation because, in the effective four-quark Lagrangian, it appears modulated by a Wilson coefficient  $C_8(\mu^2)$  which has a sizeable imaginary part induced by the Kobayashi–Maskawa phase in the flavour mixing matrix of the underlying Electroweak Model.

It is well known [12] that the bosonization of the  $Q_7$  and  $Q_8$  operators leads to a term with no derivatives in the low energy effective chiral Lagrangian. This Lagrangian generates  $|\Delta S| = 1$  transitions among the pseudoscalar Goldstone fields of the spontaneously broken  $SU(3)_L \times SU(3)_R$  symmetry of the QCD Lagrangian with three massless flavours  $u, d, s$ . It has the following form:

$$\mathcal{L}_{|\Delta S|=1}^{(0)} = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{\pi} V_{ud} V_{us}^* \frac{M_\rho^6}{16\pi^2} \mathbf{h} \operatorname{tr} \left( U \lambda_L^{(23)} U^\dagger Q_R \right) + \text{h.c.}, \quad (1.2)$$

where  $U$  denotes the  $3 \times 3$  matrix field which collects the octet of pseudoscalar Goldstone fields,  $Q_R = \operatorname{diag}[2/3, -1/3, -1/3]$  is the right-handed charge matrix associated with the electromagnetic couplings of the light quarks and  $\lambda_L^{(23)}$  is the effective left-handed flavour matrix  $\left( \lambda_L^{(23)} \right)_{ij} = \delta_{i2} \delta_{3j}$  ( $i, j = 1, 2, 3$ ). Under chiral rotations ( $V_L, V_R$ ):

$$U \rightarrow V_R U V_L^\dagger, \quad Q_R \rightarrow V_R Q_R V_R^\dagger, \quad \lambda_L^{(23)} \rightarrow V_L \lambda_L^{(23)} V_L^\dagger, \quad (1.3)$$

and the trace on the r.h.s. of Eq. (1.2) is an invariant. Actually, this is the only possible invariant which can generate  $|\Delta S| = 1$  transitions in the Standard Model to order  $\mathcal{O}(G_F \alpha)$  in the electroweak coupling and to  $\mathcal{O}(p^0)$  in the chiral expansion. With the normalization chosen in Eq. (1.2), the coupling constant  $\mathbf{h}$  is dimensionless. This constant plays a major rôle in the phenomenological analysis of  $K \rightarrow \pi\pi$  amplitudes. It is one of the basic couplings of the low energy effective electroweak Lagrangian of the Standard Model that one would like to evaluate.

The  $1/N_c$  expansion in QCD [13, 14] offers a specific non-perturbative framework for discussing the dynamics which governs low energy constants like  $\mathbf{h}$  in Eq. (1.2). It was recently shown in ref. [3] that the contribution to the constant  $\mathbf{h}$  from the  $Q_7$  and  $Q_8$  four-quark operators can be calculated to first non-trivial order in the  $1/N_c$  expansion.

At the theoretical level, the comparison between different approaches can be made by evaluating the same matrix elements of the operators  $Q_7$  and  $Q_8$  in each approach; provided of course that the

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<sup>1</sup>See refs. [1] and [2] and references therein.

calculations are made at the same scale  $\mu$  and within the same renormalization scheme. In this respect it is important to remember, as clearly emphasized in ref. [15], that it is *not* enough to just state the definition of  $\gamma_5$  in  $D$  dimensions, i.e. whether one uses 't Hooft-Veltman (HV) or naive dimensional regularization (NDR) or any other definition. One must also define the so-called evanescent operators and in this work we stick to the treatment and convention of ref. [16]. In particular we emphasize that within this convention Fierz symmetry *is* a valid concept even in  $D$  dimensions.

However, there is still a technical difficulty when comparing theoretical evaluations of matrix elements. The calculations within the dispersive approach reported in ref. [4], although aimed at evaluating matrix elements at  $\mathcal{O}(p^0)$  in the chiral expansion, are not formulated within the large- $N_c$  framework; while the lattice evaluations cannot be restricted, at least at present, to specific orders neither in the chiral expansion nor in the  $1/N_c$  expansion. Of course, if, as often claimed, the approximations within each approach are *good*, then the results should agree with each other within the estimated theoretical errors. Regretfully, the comparisons which have been made so far in the literature correspond to estimates which were obtained in *different* schemes.

As seen within the combined frameworks of the chiral expansion and the  $1/N_c$  expansion, the counting of possible contributions to matrix elements of the  $Q_7$  and  $Q_8$  operators can be summarized as follows. Denoting by  $\langle Q_{7,8} \rangle|_{\mathcal{O}(p^0)}$  the contribution to  $\mathbf{h}$  in Eq. (1.2) from  $Q_{7,8}$ , one has

$$\langle Q_7 \rangle|_{\mathcal{O}(p^0)} = \underline{\mathcal{O}(N_c)} + \mathcal{O}(N_c^0), \quad \text{and} \quad \langle Q_8 \rangle|_{\mathcal{O}(p^0)} = \underline{\mathcal{O}(N_c^2)} + \mathcal{O}(N_c^0), \quad (1.4)$$

were only the underlined contributions were calculated in ref. [3]. The contribution of  $\mathcal{O}(N_c^0)$  to  $\langle Q_8 \rangle|_{\mathcal{O}(p^0)}$  is Zweig suppressed. As we shall see, it involves the sector of scalar (pseudoscalar) Green's functions where it is hinted from various phenomenological sources that the restriction to just the leading large- $N_c$  contribution may not always be a good approximation. It is precisely this issue which prevented us in ref. [3] from giving numerical values of the matrix elements of the  $Q_8$  operator. Here we shall follow a different strategy that does not suffer from the above shortcomings, which will allow us to make predictions for the matrix elements of the  $Q_8$  operator as well.

## 2 Bosonization of $Q_7$ and $Q_8$

### 2.1 THE $Q_7$ OPERATOR

Because of its left-right vector structure, the factorized component of the operator  $Q_7$ , which is  $\mathcal{O}(N_c^2)$ , cannot contribute to order  $\mathcal{O}(p^0)$  in the low-energy effective Lagrangian. The first contribution from this operator of chiral  $\mathcal{O}(p^0)$  is at least of  $\mathcal{O}(N_c)$  in the  $1/N_c$  expansion and, formally, it can be written as follows

$$Q_7 = 6 \langle O_1(\mu) \rangle \text{tr} \left( U(x) \lambda_L^{(23)} U^\dagger(x) Q_R \right)^\dagger, \quad (2.1)$$

where

$$\langle O_1 \rangle \equiv \langle 0 | (\bar{s}_L \gamma^\mu d_L) (\bar{d}_R \gamma_\mu s_R) | 0 \rangle. \quad (2.2)$$

Matrix elements of  $Q_7$  are then given by the matrix elements of the effective operator

$$\text{tr} \left( U(x) \lambda_L^{(23)} U^\dagger(x) Q_R \right) = \frac{2}{F_0^2} K^- \pi^+ + \frac{i\sqrt{2}}{F_0^3} \left[ \bar{K}^0 \pi^- \pi^+ + \frac{1}{\sqrt{2}} K^- \pi^+ \pi^0 + \dots \right] \quad (2.3)$$

on the r.h.s. of Eq. (2.1) times the dynamical factor  $6\langle O_1 \rangle$ . In particular, defining the isospin decomposition as ( $a = 7, 8$ )

$$\begin{aligned} \langle \pi^+ \pi^- | Q_a | K^0 \rangle &= i \langle (\pi\pi)_{I=0} | Q_a | K^0 \rangle + \frac{i}{\sqrt{2}} \langle (\pi\pi)_{I=2} | Q_a | K^0 \rangle \\ \langle \pi^0 \pi^0 | Q_a | K^0 \rangle &= i \langle (\pi\pi)_{I=0} | Q_a | K^0 \rangle - i \sqrt{2} \langle (\pi\pi)_{I=2} | Q_a | K^0 \rangle, \end{aligned} \quad (2.4)$$

and taking into account that the operator of Eq. (2.3) has no transition between a  $K^0$  and a two- $\pi^0$  state, we find that

$$\langle (\pi\pi)_{I=2} | Q_7 | K^0 \rangle = \langle (\pi\pi)_{I=2} | Q_7^{(3/2)} | K^0 \rangle = \frac{-1}{F_0} \langle \pi^+ | Q_7^{(3/2)} | K^+ \rangle = \frac{-1}{3F_0} \langle \pi^+ | Q_7 | K^+ \rangle \quad (2.5)$$

$$= -\frac{4}{F_0^3} \langle O_1 \rangle, \quad (2.6)$$

where  $Q_7^{(3/2)}$  denotes the isospin  $I=3/2$  component of the  $Q_7$  operator, i.e.

$$Q_7^{(3/2)} = 2\bar{s}_L \gamma^\mu d_L (\bar{u}_R \gamma_\mu u_R - \bar{d}_R \gamma_\mu d_R) + 2\bar{s}_L \gamma^\mu u_L \bar{u}_R \gamma_\mu d_R. \quad (2.7)$$

Notice that the relations in Eq. (2.5) are exact symmetry properties of the  $\mathcal{O}(p^0)$  term in the chiral expansion.

## 2.2 THE $Q_8$ OPERATOR

As pointed out in ref. [4], independently of large- $N_c$  considerations, the bosonization of the  $Q_8$  operator to  $\mathcal{O}(p^0)$  in the chiral expansion can also be related to four-quark condensates by current algebra Ward identities, with the result

$$Q_8 = -12 \langle O_2(\mu) \rangle \text{tr} \left( U(x) \lambda_L^{(23)} U^\dagger(x) Q_R \right)^\dagger, \quad (2.8)$$

where

$$\langle O_2 \rangle \equiv \langle 0 | (\bar{s}_L s_R) (\bar{d}_R d_L) | 0 \rangle. \quad (2.9)$$

From this result, recalling Eqs. (2.4), there follows the corresponding matrix element relations

$$\langle (\pi\pi)_{I=2} | Q_8 | K^0 \rangle = \langle (\pi\pi)_{I=2} | Q_8^{(3/2)} | K^0 \rangle = \frac{-1}{F_0} \langle \pi^+ | Q_8^{(3/2)} | K^+ \rangle = \frac{-1}{3F_0} \langle \pi^+ | Q_8 | K^+ \rangle \quad (2.10)$$

$$= \frac{8}{F_0^3} \langle O_2 \rangle, \quad (2.11)$$

where, again, the relations in Eq. (2.10) are exact symmetry properties of the  $\mathcal{O}(p^0)$  term. The operator  $Q_8^{(3/2)}$  denotes the isospin  $I=3/2$  component of the  $Q_8$  operator, i.e.

$$Q_8^{(3/2)} = -4(\bar{s}_L u_R)(\bar{u}_R d_L) + 4(\bar{s}_L d_R) [(\bar{d}_R d_L) - (\bar{u}_R u_L)]. \quad (2.12)$$

## 3 The large- $N_c$ approach

### 3.1 EVALUATION OF $\langle O_1 \rangle$

As shown in ref. [3] the vacuum expectation value  $\langle O_1 \rangle$  can be expressed as an integral

$$\langle O_1 \rangle = \frac{1}{6} \left( -3i g_{\mu\nu} \int \frac{d^4 q}{(2\pi)^4} \Pi_{LR}^{\mu\nu}(q) \right) \quad (3.1)$$

involving the two-point function

$$\Pi_{LR}^{\mu\nu}(q) = 2i \int d^4 x e^{iq \cdot x} \langle 0 | T (L^\mu(x) R^\nu(0)^\dagger) | 0 \rangle, \quad (3.2)$$

with currents

$$L^\mu = \bar{q}_i(x) \gamma^\mu \frac{1 - \gamma_5}{2} q_j(x) \quad \text{and} \quad R^\mu = \bar{q}_i(x) \gamma^\mu \frac{1 + \gamma_5}{2} q_j(x), \quad (3.3)$$

and  $i$  and  $j$  fixed flavour indices,  $i \neq j$ . This is the same two-point function which governs the electroweak  $\pi^+ - \pi^0$  mass difference [17]; however, because of the factorization of the short-distance contributions in the Wilson coefficient  $C_7(\mu^2)$ , the integral that appears in Eq. (3.1) is divergent for large  $Q^2$ . Then, for consistency of the matching between the long-distance evaluation which we are concerned with here, and the short-distance evaluation, the integral in Eq. (3.1) should be done in the same renormalization scheme as the calculation of the Wilson coefficients. The fact that we are interested in  $\mathcal{O}(p^0)$  terms in the chiral Lagrangian implies furthermore that the calculation must be done in the chiral limit.

As a first step, we shall therefore use dimensional regularization with  $D = 4 - \epsilon$ , and define the integral in Eq. (3.1) as follows

$$\left( -3ig_{\mu\nu} \int \frac{d^4 q}{(2\pi)^4} \Pi_{LR}^{\mu\nu}(q) \right)_D \rightarrow \frac{3(1-D)}{16\pi^2} \frac{(4\pi\mu^2)^{\epsilon/2}}{\Gamma(2-\epsilon/2)} \int_0^\infty dQ^2 (Q^2)^{1-\epsilon/2} (-Q^2 \Pi_{LR}(Q^2))|_D, \quad (3.4)$$

where  $Q^2 = -q^2$  and

$$\Pi_{LR}^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_{LR}(Q^2). \quad (3.5)$$

The matching to short distances is controlled by the operator product expansion (OPE) [18], and in  $D$  dimensions and in the large- $N_c$  limit one has

$$\lim_{Q^2 \rightarrow \infty} (1-D) Q^4 (-Q^2 \Pi_{LR}(Q^2))|_D = -12\pi^2 \left( \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right) \left[ 1 + (\kappa - 2/3) \frac{\epsilon}{2} \right] \langle \bar{\psi}\psi \rangle^2, \quad (3.6)$$

where  $\kappa$  depends on the renormalization scheme <sup>2</sup>:

$$\kappa = -1/2 \quad \text{in NDR}, \quad \text{and} \quad \kappa = +3/2 \quad \text{in HV}. \quad (3.7)$$

The next step is the use of the hadronic representation of the spectral function associated with  $\Pi_{LR}(Q^2)$ , and this is where the large- $N_c$  limit plays an important simplifying rôle. In this limit the spectral function consists of the difference between an infinite number of narrow vector states and an infinite number of narrow axial-vector states, together with the Goldstone pion pole:

$$\frac{1}{\pi} \text{Im} \Pi_{LR}(t) = \sum_V f_V^2 M_V^2 \delta(t - M_V^2) - \sum_A f_A^2 M_A^2 \delta(t - M_A^2) - F_0^2 \delta(t). \quad (3.8)$$

This spectral function is furthermore constrained by the fact that there are no operators of dimension  $d = 2$  and  $d = 4$  which contribute to the OPE of  $\Pi_{LR}(Q^2)$ , which in turn implies the two Weinberg sum rules

$$\sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 = F_0^2 \quad \text{and} \quad \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0. \quad (3.9)$$

From the fact that  $\Pi_{LR}(Q^2)$  obeys an unsubtracted dispersion relation it then follows that

$$-Q^2 \Pi_{LR}(Q^2) = \sum_A \frac{f_A^2 M_A^6}{Q^2(Q^2 + M_A^2)} - \sum_V \frac{f_V^2 M_V^6}{Q^2(Q^2 + M_V^2)}. \quad (3.10)$$

Matching now the OPE of the function  $\Pi_{LR}(Q^2)|_D$  in the quark-gluon language, as given in Eqs. (3.6) and (3.7), onto the corresponding one in the hadronic language in Eq. (3.10) requires

$$\left( \sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 \right)_{D=4-\epsilon} = \left( \sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 \right)_{D=4} \times \left( 1 + \kappa \frac{\epsilon}{2} \right), \quad (3.11)$$

The integral in Eq. (3.4) can now be made in the hadronic representation of large- $N_c$  QCD. Using the  $\overline{\text{MS}}$ -renormalization scheme, we get the following result

$$\langle O_1 \rangle = \left( -\frac{1}{2} ig_{\mu\nu} \int \frac{d^4 q}{(2\pi)^4} \Pi_{LR}^{\mu\nu}(q) \right)_{\overline{\text{MS}}}^{\text{ren.}} = -\frac{3}{32\pi^2} \left[ \sum_A f_A^2 M_A^6 \log \frac{\Lambda^2}{M_A^2} - \sum_V f_V^2 M_V^6 \log \frac{\Lambda^2}{M_V^2} \right], \quad (3.12)$$

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<sup>2</sup>This is in agreement with the calculation reported in ref. [4].

where

$$\Lambda^2 = \mu^2 \exp(1/3 + \kappa), \quad (3.13)$$

and  $\kappa$  has been defined in Eq. (3.7). Notice that in ref. [3] another scheme was used, different from NDR and HV, in which  $\kappa = 0$ .

### 3.2 EVALUATION OF $\langle O_2 \rangle$

In the  $1/N_c$  expansion one has that

$$\langle O_2 \rangle = \frac{1}{4} [\langle \bar{\psi}\psi(\mu) \rangle^2 + \mathcal{O}(N_c^0)] , \quad (3.14)$$

and therefore  $\langle O_2 \rangle$  is directly related to the quark condensate  $\langle \bar{\psi}\psi \rangle$ , provided we restrict ourselves to the first term in this expression which is  $\mathcal{O}(N_c^2)$ . However, as already mentioned in the introduction, there are reasons to suspect that subleading terms in the  $1/N_c$  expansion involving some Green's functions with scalar (and pseudoscalar) density operators might be important, unlike those which only involve vector (and axial-vector) currents<sup>3</sup>. Consequently one should be cautious about neglecting the  $\mathcal{O}(N_c^0)$  contributions in Eq. (3.14) and consider instead

$$\langle O_2 \rangle = \frac{1}{4} \langle \bar{\psi}\psi \rangle^2 + \langle (\bar{s}_L s_R)(\bar{d}_R d_L) \rangle_c . \quad (3.15)$$

The unfactorized contribution involves Feynman diagrams which require gluon exchanges between at least two quark loops. These are the so called Zweig-suppressed contributions, which are indeed  $\mathcal{O}(N_c^0)$  in the  $1/N_c$  expansion. This ‘‘subleading’’ term is governed by a two-point function of the type  $\langle Q^2 = -q^2, , \text{ and } i, j = u, d, s \text{ with } i \neq j \rangle$

$$\Psi_{ij}(Q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T \left\{ \bar{q}_i(x) \frac{1 + \gamma_5}{2} q_i(x) \bar{q}_j(0) \frac{1 - \gamma_5}{2} q_j(0) \right\} | 0 \rangle . \quad (3.16)$$

More precisely

$$\langle (\bar{s}_L s_R)(\bar{d}_R d_L) \rangle_c = \frac{1}{i} \left( \int \frac{d^D q}{(2\pi)^D} \Psi_{ds}(Q^2) \right)_{\overline{\text{MS}}}^{\text{ren.}} , \quad (3.17)$$

where the integral should again be defined in the same renormalization scheme as the short-distance calculation of the Wilson coefficients; which explains the meaning of the  $\overline{\text{MS}}$  subscript. There is, in particular, a contribution to this integral from the singlet  $\eta^{(0)}$  pseudoscalar, which in the chiral limit acquires a mass because of the axial  $U(1)$  anomaly. This contribution, which very likely plays an important rôle in the low- $Q^2$  regime of the integral in Eq. (3.17), can easily be calculated with the result

$$\langle (\bar{s}_L s_R)(\bar{d}_R d_L) \rangle_{\eta^{(0)}} = -\frac{M_{\eta^{(0)}}^2}{16\pi^2 F_0^2} \frac{\langle \bar{\psi}\psi \rangle^2}{6} \left( \log \frac{\mu^2}{M_{\eta^{(0)}}^2} + 1 \right) , \quad (3.18)$$

in agreement with results reported in ref. [21], indicating the existence of  $\mathcal{O}(N_c^0)$  terms which could be potentially important. In this respect we wish to point out that the failure to properly incorporate  $\mathcal{O}(N_c^0)$  Zweig-suppressed contributions is also a problem which may seriously affect the *quenched* lattice calculations of matrix elements of  $Q_8$ .

To allow for the possibility of large deviations from the naive factorization in Eq. (3.14), it has become conventional to assume an ansatz of the type [22]

$$\langle O_2 \rangle(\mu) = \frac{\rho(\mu)}{4} \langle \bar{\psi}\psi \rangle^2(\mu) , \quad (3.19)$$

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<sup>3</sup>There are several phenomenological examples of this: the  $\eta'$  mass, the possible existence of a broad  $\sigma$  meson, large final state interactions in states with  $J = 0$  and  $I = 0$ , etc... [19] In order to study this issue in a systematic way we are considering at present the possibility that the appropriate expansion for these exceptional Green's functions could be a  $1/N_c$  expansion in which  $n_f/N_c$  is held fixed, where  $n_f$  denotes the number of light flavours. This kind of expansion was originally advocated by G. Veneziano [20].

where  $\rho(\mu)$  parametrizes possible deviations from the leading  $\mathcal{O}(N_c^2)$  factorization where  $\rho \rightarrow 1$ . The crucial observation [4] is that, on the one hand, the same vev  $\langle O_2 \rangle$  also appears in the OPE which governs the high- $Q^2$  behaviour of the  $\Pi_{LR}(Q^2)$  function (at  $D = 4$  and to lowest order in the perturbative evaluation of the Wilson coefficient [18]),

$$\lim_{Q^2 \rightarrow \infty} (-Q^2 \Pi_{LR}(Q^2)) Q^4 = 4\pi^2 \frac{\alpha_s}{\pi} \left( 4\langle O_2 \rangle + \frac{2}{N_c} \langle O_1 \rangle \right) + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2. \quad (3.20)$$

On the other hand, we know that large- $N_c$  QCD gives a reliable description for the  $\Pi_{LR}(Q^2)$  function since it involves the vector and axial-vector channels only. In fact, we can use the renormalization group to resum this lowest-order result into [3]

$$\lim_{Q^2 \rightarrow \infty} (-Q^6 \Pi_{LR}(Q^2)) = \frac{4\pi^2}{-\beta_1} \rho(Q) \langle \widehat{\psi\psi} \rangle^2 \left( \frac{-\beta_1 \alpha_s(Q)}{\pi} \right)^{\frac{2\gamma_1 + \beta_1}{\beta_1}}, \quad (3.21)$$

where we have neglected the term proportional to  $\langle O_1 \rangle$ , an approximation which we shall justify *a posteriori*. Here,  $\langle \widehat{\psi\psi} \rangle = \langle \bar{\psi}\psi \rangle(\mu)(\log \mu/\Lambda_{\overline{MS}})^{-9/22}$  denotes the scale invariant quark condensate and  $\rho(Q)$  also includes the effect of next-to-leading perturbative corrections. Inserting the large- $N_c$  expression which follows from Eq. (3.10) in the l.h.s. of Eq. (3.21) results (at large  $Q^2$ ) in

$$\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 \simeq \frac{4\pi^2}{\beta_1} \rho(Q) \langle \widehat{\psi\psi} \rangle^2 \left( \frac{-\beta_1 \alpha_s(Q)}{\pi} \right)^{\frac{2\gamma_1 + \beta_1}{\beta_1}}; \quad (3.22)$$

and therefore

$$\langle O_2 \rangle(\mu) \simeq \frac{1}{16\pi\alpha_s(\mu)} \left( \sum_A f_A^2 M_A^6 - \sum_V f_V^2 M_V^6 \right). \quad (3.23)$$

It is worth noticing the remarkable  $Q^2$  independence of the r.h.s. of Eq. (3.22) due to the near cancellation of the exponent  $\frac{2\gamma_1 + \beta_1}{\beta_1} = 2/11$  and the fact that  $\rho(Q) \rightarrow 1$  as  $Q^2 \rightarrow \infty$ . Since the l.h.s. is clearly  $Q$  independent, this is obviously a welcome result.

We would like to pause here and comment on the  $\mathcal{O}(\alpha_s^2)$  term in Eq. (3.20). It is our understanding that the only existing calculation in the literature of this term [23] was done in a *different* scheme of evanescent operators from the one of ref. [16]. Since this  $\mathcal{O}(\alpha_s^2)$  term is sensitive to the precise definition of evanescent operators [24]<sup>4</sup> and since matrix elements of  $Q_{7,8}$  eventually will have to be used in conjunction with Wilson coefficients to obtain a physical result, it is of the utmost importance that the extraction of  $\langle O_2 \rangle$  in Eq. (3.20) be done within the same convention for evanescent operators as that employed in the calculation of Wilson coefficients. This is why we think, unlike the authors of refs. [4, 5], that it is misleading to use the existing calculation of the  $\mathcal{O}(\alpha_s^2)$  term in Eq. (3.20) and, consequently, we refrain from including it to obtain  $\langle O_2 \rangle$  in Eq. (3.23). This fact immediately introduces a major difference with respect to refs. [4, 5] as the contribution of  $\mathcal{O}(\alpha_s^2)$  they take turns out to be large, namely  $\sim 50\%$ .

Numerical estimates for  $\langle O_1 \rangle$  in Eq. (3.12) and  $\langle O_2 \rangle$  in Eq. (3.23) will be made in the last section using the *minimal hadronic ansatz* approximation to large- $N_c$  QCD (MHA). This approximation consists in limiting the large- $N_c$  spectrum of narrow states to the minimal number required to satisfy the OPE constraints relevant to the process that one is considering. In our case, this requires the consideration of one vector state and one axial-vector state, besides the pion pole<sup>5</sup>. The MHA approximation has recently been shown [26] to successfully reproduce sum rules which use the LEP experimental data of the ALEPH collaboration on hadronic  $\tau$  decay. This leads to a contribution of  $\langle O_1 \rangle$  to the r.h.s. of Eq. (3.20) which is indeed much smaller than the one obtained from  $\langle O_2 \rangle$  in either scheme NDR or HV, justifying *a posteriori* the initial approximation which was made to obtain Eq. (3.23).

<sup>4</sup>The analysis of ref. [24] shows that this term of  $\mathcal{O}(\alpha_s^2)$  can actually be reduced in half by a redefinition of the operator basis in the  $4 - D$  extra dimensions.

<sup>5</sup>Sometimes we have referred to this particular case as the *lowest meson dominance* approximation (LMD) [25].

#### 4 The dispersive approach and the rôle of “higher dimension operators”

When computing  $\langle O_1 \rangle$  through Eq. (3.1), the authors of ref. [4] have chosen to split the integral in the regions  $0 \leq Q^2 \leq \mu^2$  and  $\mu^2 \leq Q^2 \leq \infty$  where  $\mu$  plays the role of a sharp momentum cutoff, i.e. not to be confused with the scale appearing in dimensional regularization. The low- $Q^2$  part is amenable to the use of the experimental data through a dispersion relation. The high- $Q^2$  part, on the other hand, is divergent and must be renormalized, e.g., in dimensional regularization in  $D = 4 - \epsilon$  dimensions with minimal subtraction ( $\overline{MS}$ ). In this way one obtains

$$\langle O_1 \rangle^{\overline{MS}}(\mu) = \langle O_1 \rangle^{\text{cutoff}}(\mu) + \frac{(1-D)}{32\pi^2} \frac{(4\pi\mu^2)^{\epsilon/2}}{\Gamma(2-\epsilon/2)} \int_{\mu^2}^{\infty} dQ^2 (Q^2)^{1-\epsilon/2} (-Q^2 \Pi_{LR}(Q^2)) , \quad (4.1)$$

where  $\langle O_1 \rangle^{\text{cutoff}}(\mu)$  corresponds to the low- $Q^2$  part, i.e.

$$\langle O_1 \rangle^{\text{cutoff}}(\mu) = \frac{3}{32\pi^2} \int_0^{\infty} dt t^2 \log \frac{t + \mu^2}{t} \frac{1}{\pi} \text{Im} \Pi_{LR}(t) . \quad (4.2)$$

Notice that, when writing Eq. (4.2), one is using the first and second Weinberg sum rules.

We are now in the position to discuss the issue of “higher dimension operators” in the calculation of weak matrix elements which has been recently raised in refs. [10, 11]. On the one hand, these authors have stressed the fact that the long-distance evaluation of weak matrix elements should be done in the same renormalization scheme as the short-distance evaluation of the Wilson coefficients. Although this is a well known fact, they observe, quite rightly, that most of the available long-distance calculations of weak matrix elements are done in a cut-off scheme while using the  $\overline{MS}$  results of the Wilson coefficients. Exceptions to that are the lattice QCD simulations and the large- $N_c$  analytic results in refs. [1, 2]. On the other hand, a more subtle issue also raised in refs. [10, 11] is the fact that in doing integrals of Green’s functions which govern the long-distance evaluation of weak matrix elements, in the same  $\overline{MS}$  renormalization as used for the Wilson coefficients, one is still confronted with the fact that, in principle, the full string of terms in the OPE contribute to these integrals because the integration over the euclidean momentum goes all the way to infinity. We entirely agree with this observation but we want to show how the large- $N_c$  approach we have been advocating avoids this criticism while the dispersive approach, as used in ref. [4], fails to incorporate the effect of higher dimension operators.

We consider first the evaluation of the vev  $\langle O_1 \rangle^{\overline{MS}}$  in Eq. (4.1) to illustrate the point. Depending on the input for  $\Pi_{LR}$  in the integral over the high- $Q^2$  region in this equation, one may obtain a unsatisfactory determination of  $\langle O_1 \rangle^{\overline{MS}}$  even if  $\langle O_1 \rangle^{\text{cutoff}}$  is derived reliably from the experimental data. The authors of ref. [4] input for  $\Pi_{LR}$  its *leading* asymptotic behaviour coming from dimension six operators in the OPE, i.e. the equivalent to our large- $N_c$  Eq. (3.6). While for  $\mu$  large enough this is certainly a reasonable thing to do, for the realistic value of  $\mu = 2$  GeV the question still remains as to how large the contribution from higher dimension operators in the OPE may actually get to be [10, 11].

In this regard we would like to point out that, in the large- $N_c$  approach, the integrand in Eq. (3.10), if expanded in powers of  $1/Q^2$  gives not only the leading power of the OPE, which we use as a constraint, but also the higher powers. As a matter of fact, and this is the crucial point, in ref. [26] it is shown that the MHA approximation to large- $N_c$  discussed at the end of section (3) does a remarkably good job in predicting these higher vev’s as compared to the ALEPH data. Therefore one can use the MHA approximation to confidently evaluate the size of the contribution from higher dimension operators to the high- $Q^2$  region in the integral of Eq. (4.1). One then obtains

$$\langle O_1 \rangle^{\overline{MS}}(\mu) = \langle O_1 \rangle^{\text{cutoff}}(\mu) - \frac{3}{2\pi} \alpha_s \langle O_2 \rangle \left( \frac{1}{3} + \kappa \right) \quad (4.3)$$

$$- \frac{3}{32\pi^2} \left( \sum_V f_V^2 M_V^6 \log \frac{\mu^2 + M_V^2}{\mu^2} - \sum_A f_A^2 M_A^6 \log \frac{\mu^2 + M_A^2}{\mu^2} \right) , \quad (4.4)$$



where the expression (4.4) reads in the MHA approximation ( $g_A = M_V^2/M_A^2$ )

$$\simeq -\frac{3}{32\pi^2} \frac{F_0^2 M_V^4}{1-g_A} \left( \log \frac{\mu^2 + M_V^2}{\mu^2} - \frac{1}{g_A} \log \frac{\mu^2 + \frac{M_V^2}{g_A}}{\mu^2} \right). \quad (4.5)$$

Equation (4.3) is the result of ref. [4] (in our notation), with only the contribution from dimension-six operators. The expression in Eq. (4.5), which represents the contribution from operators of dimension 8 and higher, indeed goes to zero as an inverse power for large values of  $\mu$ , as expected. However it amounts to a  $\sim 50\%$  reduction at  $\mu = 2$  GeV in NDR (see next section). We emphasize that in ref. [26] the MHA approximation was shown to yield correct predictions in the OPE up to operators of dimension 10, so we consider Eq. (4.5) to be reliable at least up to this order.

The corresponding discussion in the case of  $\langle O_2 \rangle$  starts at Eq. (3.20) with the contribution from  $\langle O_1 \rangle$  neglected, as already explained. Therefore  $\langle O_2 \rangle$  is the operator modulating the  $1/Q^6$  fall-off in the OPE of the function  $\Pi_{LR}$ . At large  $N_c$  one obtains

$$\langle O_2 \rangle \approx \frac{C(Q^2)}{16\pi\alpha_s(\mu)} [-Q^6 \Pi_{LR}(Q^2)] , \quad (4.6)$$

where

$$\begin{aligned} C(Q^2) &= \left[ 1 - \frac{\sum_A \frac{f_A^2 M_A^8}{Q^2 + M_A^2} - \sum_V \frac{f_V^2 M_V^8}{Q^2 + M_V^2}}{\sum_A f_A^2 M_A^6 - \sum_V f_V^2 M_V^6} \right]^{-1} \\ &\simeq \left[ 1 - \frac{M_V^2}{Q^2 + M_V^2} \frac{Q^2(g_A + 1) + M_V^2}{g_A Q^2 + M_V^2} \right]^{-1} , \end{aligned} \quad (4.7)$$

where the last expression is a consequence of the MHA approximation. Again  $C(Q^2) - 1$  vanishes as an inverse power at high  $Q^2$ . Since in refs. [4, 5] the contribution from higher dimension operators is neglected, these authors effectively take  $C(Q^2) = 1$ . However a typical value for  $C(Q^2)$  is  $\sim 40\%$  above unity for  $Q = 2$  GeV (see next section for details).

## 5 Numerical Estimates and Comparisons

Using Eqs. (3.12), (3.13), (3.7) and (3.23) one obtains in the MHA approximation to large- $N_c$  QCD the simple expressions

$$\langle O_1 \rangle(\mu) = -\frac{3}{32\pi^2} \frac{F_0^2 M_V^4}{1-g_A} \log \left[ g_A \left( \frac{g_A \Lambda^2}{M_V^2} \right)^{\frac{1}{g_A}-1} \right] \quad (5.1)$$

$$\langle O_2 \rangle(\mu) = \frac{1}{16\pi\alpha_s(\mu)} \frac{F_0^2 M_V^4}{g_A} . \quad (5.2)$$

For their numerical evaluation we shall take [26]

$$M_V = (750 \pm 25) \text{ MeV}, \quad F_0 = (87 \pm 3) \text{ MeV}, \quad g_A = 0.50 \pm 0.06, \quad (5.3)$$

and  $\alpha_s(\mu=2 \text{ GeV}) = 0.33 \pm 0.04$ . Then the previous expressions lead to

$$\langle O_1 \rangle_{(\mu=2 \text{ GeV})} = \left. \begin{aligned} &(-1.9 \pm 0.2) \times 10^{-5} \text{ GeV}^6 \text{ in NDR} \\ &(-1.1 \pm 0.2) \times 10^{-4} \text{ GeV}^6 \text{ in HV} \end{aligned} \right\} \quad (5.4)$$

$$\langle O_2 \rangle_{(\mu=2 \text{ GeV})} = (2.9 \pm 0.6) \times 10^{-4} \text{ GeV}^6 \text{ both in NDR and HV}, \quad (5.5)$$

which in turn imply, through Eqs. (2.5), (2.6), (2.10) and (2.11),

$$\langle(\pi\pi)_{I=2}|Q_7|K^0\rangle_{(\mu=2\text{ GeV})} = \left. \begin{array}{ll} (0.11 \pm 0.01) \text{ GeV}^3 & \text{in NDR} \\ (0.67 \pm 0.09) \text{ GeV}^3 & \text{in HV} \end{array} \right\}, \quad (5.6)$$

and

$$\langle(\pi\pi)_{I=2}|Q_8|K^0\rangle_{(\mu=2\text{ GeV})} = (3.5 \pm 0.8) \text{ GeV}^3. \quad (5.7)$$

In order to facilitate the direct comparison with the lattice results we also give the equivalent result for the matrix elements

$$\langle\pi^+|Q_7^{(3/2)}|K^+\rangle_{(\mu=2\text{ GeV})} = \left. \begin{array}{ll} (-9.8 \pm 0.6) \times 10^{-3} \text{ GeV}^4 & \text{in NDR} \\ (-5.8 \pm 0.8) \times 10^{-2} \text{ GeV}^4 & \text{in HV} \end{array} \right\}, \quad (5.8)$$

and

$$\langle\pi^+|Q_8^{(3/2)}|K^+\rangle_{(\mu=2\text{ GeV})} = (-0.30 \pm 0.07) \text{ GeV}^4. \quad (5.9)$$

These are our predictions for the  $O_{1,2}$  condensates and for the  $Q_{7,8}$  matrix elements. The errors quoted come only from the propagation from the input values for  $M_V$ ,  $F_0$ ,  $g_A$  and  $\alpha_s(2\text{ GeV})$ . We remark that, unlike the case of  $Q_7$ , matrix elements of  $Q_8$  do not show any dependence on the renormalization scheme (NDR vs. HV) at this level.

In Table 1 we give a joint comparison of our results with the existing different evaluations of matrix elements with which we can compare scheme dependences explicitly <sup>6</sup>.

Table 1: Summary of matrix elements  $M_{7,8} \equiv \langle(\pi\pi)_{I=2}|Q_{7,8}|K^0\rangle_{(2\text{ GeV})}$  using naive dimensional regularization (NDR) and the 't Hooft-Veltman scheme (HV), in units of  $\text{GeV}^3$ .

Matrix Elements	$M_7(\text{NDR})$	$M_7(\text{HV})$	$M_8(\text{NDR})$	$M_8(\text{HV})$
refs. [4, 5]	$0.22 \pm 0.05$		$1.3 \pm 0.3$	
ref. [6]	$0.11 \pm 0.04$	$0.18 \pm 0.06$	$0.51 \pm 0.10$	$0.62 \pm 0.12$
This work (see also ref. [3])	$0.11 \pm 0.03$	$0.67 \pm 0.20$	$3.5 \pm 1.1$	$3.5 \pm 1.1$

In the table we have rounded off the errors of our predictions to an overall 30%, which we believe to be a generous estimate of the systematic errors in our approach [30]. For the case of  $M_8$  this has the caveat that it of course assumes that the  $\mathcal{O}(\alpha_s^2)$  corrections to Eq. (3.20), once computed in the right operator basis [16], will be of a reasonable size.

We find that, within the combined errors, our results for  $M_7$  are in agreement with the lattice results in the NDR scheme, but not in the HV scheme<sup>7</sup>; while they disagree by a factor of two with the dispersive results. The origin of this disagreement can in fact be traced back to the effect of higher terms in the OPE found in Eq. (4.5) which, as discussed in the previous section, have been ignored in the dispersive approach. As to the results of  $M_8$ , we are in disagreement with both the lattice and the dispersive results. The discrepancy with the former may originate in the fact that most of the contribution comes from an OZI-violating Green's function which is something inaccessible in the quenched approximation. As a matter of fact, lattice results are compatible with a value  $\rho \sim 1$  in Eq. (3.19) whereas we are finding that  $\rho \sim 6$  for  $\langle\bar{\psi}\psi\rangle(2\text{ GeV}) \sim (-0.240\text{ GeV})^3$ , which is also the phenomenological result found in ref. [27]. On the other hand, we do agree with the lattice results on the independence of scheme (NDR vs. HV) in  $M_8$ . As to the discrepancy on  $M_8$  with the dispersive results, it can be traced back to the  $\sim 50\%$   $\mathcal{O}(\alpha_s^2)$  correction they use for the Wilson coefficient in Eq. (3.20) plus the  $\sim 40\%$  effect coming from the contribution from higher dimension operators in Eq. (4.7).

<sup>6</sup>This explains why, in this table, we only quote the lattice results of ref. [6]. Model dependent calculations of the so called  $B$  factors associated with the  $Q_{7,8}$  operators can also be found, in order of increasing sophistication, in refs. [28], [21] and [29].

<sup>7</sup>This could be due to the difference in what the Rome and Munich groups actually call the ‘‘HV scheme’’ in ref. [16].

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